Exercise 6

In Exercises 5-8, derive the general solution of the given equation by using an appropriate change of variables, as we did in Example 3.

$$2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0.$$

Solution

Make the change of variables, $\alpha = 2x + 3t$ and $\beta = 2x - 3t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(2) + \frac{\partial u}{\partial \beta}(2) = 2\frac{\partial u}{\partial \alpha} + 2\frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(3) + \frac{\partial u}{\partial \beta}(-3) = 3\frac{\partial u}{\partial \alpha} - 3\frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 0 &= 2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} \\ &= 2\left(3\frac{\partial u}{\partial \alpha} - 3\frac{\partial u}{\partial \beta}\right) + 3\left(2\frac{\partial u}{\partial \alpha} + 2\frac{\partial u}{\partial \beta}\right) \\ &= 12\frac{\partial u}{\partial \alpha}. \end{aligned}$$

Divide both sides by 12.

$$\frac{\partial u}{\partial \alpha} = 0$$

Integrate both sides partially with respect to α to get u.

$$u(\alpha, \beta) = f(\beta)$$

Here f is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x, t) = f(2x - 3t)$$